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Oxtoby's pseudocompleteness revisited

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Abstract

It is well known that the class of Baire spaces is not productive, and the Baire Unification problem calls for finding subclasses that are productive and have "nice" inheritance properties. This paper extends a modified version of the pseudocompleteness property of J.C. Oxtoby using the bitopological spaces of J.C. Kelly, and considers its relation to the Baire property and (non) permanence properties including productivity and inheritance by G_δ subsets of both.

Keywords: Baire space; Pseudocomplete space; Oxtoby space; Bitopological space; Quasimetric space; Quasiregular space; π^0-base; Π-related topologies; G_δ-subspace; σ-product; Countably subcompact; Almost locally compact

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1. Introduction

Recall that a topological space in which a countable intersection of dense open sets is dense is called a *Baire space*. It has long been known that the class **Ba** of Baire spaces includes all locally compact Hausdorff spaces and all completely metrizable spaces, as well as all products of such spaces. Moreover, every open subspace and every dense G_δ of a Baire space is a Baire space, but the non-Baire space $\mathbb{Q}$ of rational numbers is a

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dense subspace of the Baire space $\mathbb{R}$ of real numbers. As shown by Oxtoby [15] under the continuum hypothesis, the class **Ba** fails to be productive; indeed, there is a metrizable Baire space whose topological product with itself fails to be a Baire space. (For in [5], a Baire space X is exhibited such that X^2 is not a Baire space, and in [13], a map # from topological spaces to metric spaces is given such that for any space Y, $X \times Y$ is Baire if and only if $(\#X) \times Y$ is Baire. Thus, $\#X$ is a metric Baire space whose square is not a Baire space.)

Because of such aberrant behavior of **Ba**, special classes of Baire spaces have been considered. In [3], the authors seek natural classes of Baire spaces that include all locally compact Hausdorff and completely metrizable spaces, and call this search the *Unification problem.* In this paper a modified version of the pseudocompleteness property of Oxtoby (see [15]) is studied. Its definition will be recalled shortly after some preliminary concepts have been reviewed. Part of our objective is to is to reduce the extent to which separation axioms are used.

Throughout, we use the following notations: If $m \leqslant \omega$, then $(y_n)_{n<m}$ represents a sequence of *length m*; in particular, if $0 < m < \omega$, $(y_n)_{n<m} = (y_0, \ldots, y_{m-1})$, and we adopt the conventions $(y_n)_{n<0} = \emptyset$ and $(y_n) = (y_n)_{n<\omega} = (y_0, y_1, \ldots)$. For a collection $\mathcal{K}$ of sets, we also let $\mathcal{K}^+$ denote $\mathcal{K} \setminus \{\emptyset\}$, the collection of its nonempty elements.

Definition 1.1. Given a topology τ and $m \leqslant \omega$, a *nest of length m* is a decreasing sequence $(B_n)_{n<m}$ of sets with nonempty interiors (that is, a sequence $(B_n)_{n<m}$, such that if $n < m$ then $\operatorname{int} B_n \neq \emptyset$, and if $n + 1 < m$, then $B_n \supset B_{n+1}$). By a *strong nest* is meant a nest $(B_n)_{n<m}$ such that $\operatorname{int} B_n \supset \operatorname{cl} B_{n+1}$ whenever $n + 1 < m$.

A *pseudobase* (or π*-base*) for a topology τ is a collection of nonempty open sets such that each nonempty open set contains one of them.

A collection $\mathcal{B}$ of subsets of a topological space (X, τ) is a π^0*-base* for τ if:

(i) each $B \in \mathcal{B}$ has nonempty interior, and

(ii) for each nonempty open U, there is a subset of U which is in $\mathcal{B}$.

Thus, the interiors of sets in a π^0-base for a topology form a π-base for it. Todd in [19] used the term pseudobase for what we now call a π^0-base.

Recall that a topological space (X, τ) is *quasiregular* if each $U \in \tau^+$ contains $\operatorname{cl} V$ for some $V \in \tau^+$.

A sequence $(\mathcal{B}_n)_{n<m}$ is an *associated* nest for a sequence $(\mathcal{B}_n)_{n<m}$ of π^0-bases for τ provided that for each $n < m$, $B_n \in \mathcal{B}_n$. An *Oxtoby sequence* for τ is an infinite sequence of π^0-bases for which each associated nest has nonempty intersection. A topological space (X, τ) is an *Oxtoby space* if it has an Oxtoby sequence, and it is *pseudocomplete* if it is quasiregular and has an infinite sequence $(\mathcal{B}_n)$ of π^0-bases for which each associated strong nest (B_n) has nonempty intersection.

Notice that no separation property (including the rarely used quasiregularity) is needed to show the following:

Theorem 1.2. *Suppose (X, τ) is a topological space.*

(a) *If* (X, τ) *is an Oxtoby space, then it is a Baire space.*
(b) *A* π^0*-base for a dense subspace* Y *of* X *is the restriction of a* π^0*-base for* X*.*
(c) (X, τ) *is an Oxtoby space if it has a dense subspace* Y *that is an Oxtoby space.*

Proof. (a) Given a sequence (U_n) of dense open sets, and a nonempty open set U in a space (X, τ) which has an Oxtoby sequence $(\mathcal{B}_n)$, define inductively an associated nest $(B_k)_{k<\omega}$ such that:

$$U \cap U_j \cap B_j \supset B_k \text{ whenever } j < k < \omega.$$

The induction follows as the open set $U \cap \operatorname{int}(B_k)$ is nonempty, the open set U_k is dense, and $\mathcal{B}_{k+1}$ is a π^0-base. Since $(\mathcal{B}_n)$ is an Oxtoby sequence $U \cap \bigcap_n U_n \supset \bigcap_n B_n \neq \emptyset$, and so $\bigcap_n U_n$ is dense as required.

(b) Let $\mathcal{C}$ be a π^0-base of Y. Let

$$\mathcal{B} = \{V \cup C\colon\ V \in \tau,\ C \in \mathcal{C},\ V \cap Y = \operatorname{int}_{\tau|Y} C\}.$$

From the definition of $\mathcal{B}$ and a brief calculation, the trace of each member of $\mathcal{B}$ is the corresponding member of $\mathcal{C}$. Moreover, for each member C of $\mathcal{C}$ there is $V \in \tau$ with $V \cap Y = \operatorname{int}_{\tau|Y} C$, so $B = V \cup C$ is in $\mathcal{B}$. Thus $\mathcal{C}$ consists of all traces of members of $\mathcal{B}$. As each $C \in \mathcal{C}$ has nonempty $\tau|Y$-interior, each member of $\mathcal{B}$ has nonempty τ-interior. Let U be in τ^+. As Y is dense, $U \cap Y$ is in $\tau|Y^+$, and there is $C \in \mathcal{C}$ with $U \cap Y \supset C$. There is $W \in \tau$ with $W \cap Y = \operatorname{int}_{\tau|Y} C$. Let $V = U \cap W$, and note that $U \supset V \cup C$. Now $V \cap Y = U \cap W \cap Y = U \cap \operatorname{int}_{\tau|Y} C = \operatorname{int}_{\tau|Y} C$, so $V \cup C \in \mathcal{B}$. Therefore $\mathcal{B}$ is a π^0-base as required.

(c) Let $(\mathcal{C}_n)$ be an Oxtoby sequence for Y, and let each $\mathcal{B}_n$ be a π^0-base arising from $\mathcal{C}_n$ as in (b). We need only show that any nest (B_n) associated with $(\mathcal{B}_n)$ has a nonempty intersection. Each $C_n = B_n \cap Y$ is in $\mathcal{C}_n$, and, since it is a decreasing sequence (C_n) is a nest associated with the Oxtoby sequence $(\mathcal{C}_n)$. Therefore

$$\bigcap_n B_n \supset \bigcap_n C_n \neq \emptyset,$$

and so $(\mathcal{B}_n)$ is an Oxtoby sequence for (X, τ). $\square$

It turns out that introducing a second topology is helpful in analyzing the role of separation properties. This motivates us in Section 2, to define and characterize pseudocomplete bitopological spaces. We apply this characterization in a number of ways; for example to giving conditions which cause a quasimetric space to be a Baire space.

Section 3 continues the study of pseudocomplete spaces and includes results on product spaces and G_δ-subspaces. Products of pseudocomplete bitopological spaces are pseudocomplete. The concepts of two Π-related topologies and of a present subspace are used to unify, improve and apply a number of results on when pseudocompleteness is inherited by a G_δ-subspace. Corollaries include that G_δ-subspaces of the Sorgenfrey line are pseudocomplete, so Baire, and that [16] G_δ-subspaces of regular countably subcompact spaces are similarly Baire, and the well-known result that Čech complete spaces are Baire spaces.

2. A bitopological approach to separation in pseudocompleteness

A *bitopological space* is a set with an ordered pair of topologies, $\boldsymbol{X} = (X, \tau, \tau^*)$; its *dual* is $\boldsymbol{X}^* = (X, \tau^*, \tau)$. A *subspace* of $\boldsymbol{X}$ is $(Y, \tau|Y, \tau^*|Y)$, for some $Y \subset X$. It is natural to call the self-dual bitopological spaces *symmetric*, and identify a topological space (X, τ) with its *associated bitopological space*, the symmetric (X, τ, τ).

A function $d : X \times X \to [0, +\infty)$ is a *quasimetric* [4] on a set X if:

(i) for all $x \in X$, $d(x, x) = 0$, and

(ii) for all $x, y, z \in X$, $d(x, z) \leqslant d(x, y) + d(y, z)$.

If d is a quasimetric on X, then by the *topology arising from d* we mean τ_d, the topology generated by the *open d-balls*

$$B_\delta(x) = \{y \in X \colon d(x, y) < \delta\}, \quad \text{for } \delta > 0,\ x \in X.$$

By the *bitopological space arising from d* we mean $\boldsymbol{X}_d = (X, \tau_d, \tau_{d^*})$, where $d^*(x, y) = d(y, x)$. If d is symmetric, then so is $\boldsymbol{X}_d$. A *quasimetric space* (X, d) is a set X together with a quasimetric d on X, and a topological space (X, τ) is *quasimetrizable* if there is a quasimetric d on X such that $\tau = \tau_d$.

Note that the open balls $B_\delta(x)$, $\delta > 0$, give an open neighborhood base at x for the first countable topological space (X, τ_d). We call a set $C_\delta(x) = \{y \in X \colon d(x, y) \leqslant \delta\}$ a *closed d-ball*; these are closed in τ_{d^*}, not τ_d (see Proposition 2.3, Example 2.1 below). Following Kelly [14], we call a sequence $x_0, x_1, \ldots \in X$ *d-Cauchy* if, for all $\delta > 0$, there is a p such that for all $m > n > p$, $d(x_m, x_n) < \delta$; and we say that X is *d-complete*, if each d-Cauchy sequence converges.

The following is a standard and fundamental example of a quasimetric space.

Example 2.1. Let I be the unit interval [0, 1] with the topologies $\mathcal{L} = \{I \cap (-\infty, a) \colon a \in \mathbb{R}\}$ and $\mathcal{U} = \{I \cap (a, +\infty) \colon a \in \mathbb{R}\}$, here called, respectively, the lower and upper topologies on the unit interval. For the quasimetric $d(x, y) = \max\{0, y - x\}$ on $[0, 1]$, $\mathcal{L} = \tau_d$ and $\mathcal{U} = \tau_{d^*}$. Note that a nonempty member of either topology is dense for that topology. The usual topology for I is the join of these two; equivalently, it is the topology arising from the *symmetrization of* d, $d^S = d + d^*$. The symmetrization is a pseudometric for each quasimetric d, and in this case is the usual metric on I.

The Baire unification problem is one of the first in topology to which asymmetric methods were applied, due to the pioneering efforts of de Groot, Aarts and Lutzer, and others on cocompactness. (See [1–3], for example.) By and large, we use the notation of [12] for bitopological spaces; Reilly [17] has a useful bibliography of work on such spaces.

Though several properties are shared by Baire spaces and pseudocomplete spaces, there are significant differences. Products of pseudocomplete spaces are pseudocomplete, while products of Baire spaces need not be. There is an extensive literature on the latter. (See, for example, [5] for a brief and helpful survey.)

[4] Some authors call what we define a “quasi-pseudometric”.

To give the analogues of the classical Baire category theorem that first approach, we need a basic separation axiom:

Definition 2.2 [12,14]. A bitopological space (X, τ, τ^*) is *regular* if each point of X has a τ-neighborhood base consisting of τ^*-closed sets.

If the two topologies are the same, this definition is equivalent to regularity of that topology. The following is left for the reader to prove.

Proposition 2.3. *For a quasimetric d on a set X, the closed d-balls are closed in τ_{d^*}. Thus, the bitopological space arising from a quasimetric is regular.*

For us, a compact space need not be a Hausdorff space; we only require that every open cover have a finite subcover. [5] The following observations led us to bitopology:

If (X, τ, τ^*) is regular and (X, τ^*) is compact, then (X, τ) is a Baire space [8].

If (X, d) is a d^*-complete quasimetric space then (X, τ_d) is a Baire space [14]. [6]

We improve these results in Theorems 2.8 and 2.9 below. The following example [9, 5.4] shows that these results are essentially bitopological: properties of the second topology and its relationship to the first are essential to yield that the first topology is Baire.

Example 2.4. Let $X = \mathbb{Q} \cap [0, 1)$, $\mathcal{L}$ be the inherited lower topology, $\mathcal{U}$, the inherited upper topology using the bitopology for $I = [0, 1]$ defined in Example 2.1. Now $(X, \mathcal{U})$ is compact (since the only open set containing 0 is X), and X is d^*-complete ($d^*(0, x) = 0$ for each $x \in X$, so 0 is a limit for each sequence in X), $(X, \mathcal{L}, \mathcal{U})$ is regular, and thus by either of the above results, $(X, \mathcal{L})$ is Baire. But $\mathcal{L} = \tau_d$ is neither compact nor d-complete, and $(X, \mathcal{U})$ is not Baire. (Notice, by the way, that X is also countable and no point is open.)

We use the convention that int and cl are the interior and closure with respect to τ, whereas int* and cl* are taken with respect to τ^*. Further, “open”, “closed”, etc., are meant with respect to τ, while “*-open”, “*-closed” are with respect to τ^*. Note that there is an emphasis on the first topology of the bitopological space, and that separation becomes a relationship between the two topologies.

Definition 2.5. Given a bitopological space $\boldsymbol{X} = (X, \tau, \tau^*)$:

A collection of sets is a *π-base* or *π^0-base* for $\boldsymbol{X}$ if it is one for τ; a sequence of sets is a *nest* for $\boldsymbol{X}$ if it is one for τ.

A *strong nest* for $\boldsymbol{X}$ is a nest $(B_n)_{n<m}$, $m \leqslant \omega$, such that for all n with $n + 1 < m$, $\operatorname{int} B_n \supset \operatorname{cl}^* B_{n+1}$.

$\boldsymbol{X}$ is *quasiregular* if each nonempty τ-open set contains the τ^*-closure of a nonempty τ-open set.

[5] The term quasicompact is often used for such spaces; e.g., in [7,9].

[6] [14] only asserts that a countable intersection of dense open sets of τ_d is nonempty, but the proof supports this stronger conclusion.

Associated nests are defined as in Definition 1.1, and an *Oxtoby sequence* for $\boldsymbol{X}$ is one for (X, τ), while $\boldsymbol{X}$ is an *Oxtoby space* if it has an Oxtoby sequence.

$\boldsymbol{X}$ is *pseudocomplete* if it is quasiregular and there is an infinite sequence of π^0-bases for $\boldsymbol{X}$ for which each associated strong nest has nonempty intersection.

Thus quasiregularity is equivalent to the existence of a π^0-base for τ consisting of τ^*-closed sets; regularity is equivalent to the existence of a base for τ consisting of τ^*-closed sets, and so is stronger. Note that a topological space (X, τ) is quasiregular or pseudocomplete if and only if its associated bitopological space (X, τ, τ) is. The following characterization of pseudocompleteness for bitopological and topological spaces simplifies its use:

Characterization Theorem 2.6. *A bitopological* (*or topological*) *space is pseudocomplete if and only if it is a quasiregular Oxtoby space.*

A topological space (X, τ) is an Oxtoby space if and only if there is a second topology τ^* on X such that (X, τ, τ^*) is pseudocomplete (*and regular*).

Proof. Suppose $(\mathcal{B}_n)$ is an Oxtoby sequence for $\boldsymbol{X}$. Each associated strong nest is an associated nest, thus has nonempty intersection. So if, further, $\boldsymbol{X}$ is quasiregular then it is pseudocomplete by definition.

Conversely, let $\boldsymbol{X}$ be pseudocomplete; thus there is a sequence $(\mathcal{C}_n)$ of π^0-bases for τ for which each associated strong nest has nonempty intersection. Define $(\mathcal{B}_n)$ by

$$\mathcal{B}_n = \begin{cases} \{\mathrm{cl}^* C\colon\ C \in \mathcal{C}_k\} & \text{for } n = 3k-2, \\ \mathcal{C}_k & \text{for } n = 3k-1, \\ \{\mathrm{int}\, C\colon\ C \in \mathcal{C}_k\} & \text{for } n = 3k. \end{cases}$$

Then for each associated nest (B_n), (B_{3n}) is an associated strong nest for $(\mathcal{C}_n)$ and thus

$$\bigcap_{n<\omega}(B_n) = \bigcap_{n<\omega}(B_{3n}) \neq \emptyset,$$

as required. The topological statement is then the symmetric special case, $\boldsymbol{X} = (X, \tau, \tau)$.

For the last assertion, suppose (X, τ) has an Oxtoby sequence. Let τ^A be the topology generated by the τ-closed sets. Then each τ-open set is τ^A-closed, so τ certainly has a base of τ^A-closed sets. Therefore, by the first statement, (X, τ, τ^A) is pseudocomplete. □

The topology τ^A introduced to in the above proof has been widely discussed (for example, in [10,7]); in [12, 4.3], it is called the *Alexandroff dual* of τ, and additional separation properties of (X, τ, τ^A) are noted.

The next two examples are of Oxtoby spaces that are not quasiregular spaces (thus, not pseudocomplete). The second of these examples is a Hausdorff space, and is based on Example 2.7 of [19]. Of course, by Theorem 2.6 each Oxtoby topology is the initial topology of a pseudocomplete bitopological space.

Examples 2.7. (a) $\boldsymbol{X} = ([0,1), \mathcal{L})$, $\mathcal{L}$ the lower topology on X, satisfies the definition of pseudocompleteness except for the requirement of quasiregularity. In fact, if each $\mathcal{B}_n$ is $\mathcal{L}\backslash\{\emptyset\}$, then $(\mathcal{B}_n)$ is an Oxtoby sequence, but quasiregularity is ruled out by the fact that each nonempty open set is dense.

(b) Let $X = [0,1]$, τ be the usual topology, S be the set of irrationals in X, and σ be the larger topology $\{U \cup (V \cap S)\colon\ U, V \in \tau\}$. By [11, 6.K], there is a complete metric d for S, giving its subspace topology $\tau|S = \sigma|S$. $\mathcal{B}_n = \{B_\delta(c)\colon\ c \in S,\ 0 < \delta < 1/(n+1)\}$ is an Oxtoby sequence for σ since it is one for $(S, \sigma|S)$, and S is dense and open in (X, σ). Now if $W = (U \cup V) \cap S$ is a nonempty σ-open set, then $U \cup (V \cap S)$ is a nonempty σ-open subset of W. The τ-closure of any nonempty σ-open subset of $(U \cup V) \cap S$ contains elements of $X \setminus S$. For each rational $r \in X$, a neighborhood base at r for τ will also be one for σ, so its σ-closure contains the same elements of $X \setminus S$. Therefore (X, σ) is not quasiregular even though it has an Oxtoby sequence and a dense open subspace that is pseudocomplete.

We now show the results that drew our attention to pseudocompleteness. In the process, we show how quasiregularity may be used to prove that the topological space has an Oxtoby sequence. In fact, we generalize beyond the asymmetric analogue of the classical result on locally compact spaces. A quasiregular bitopological space is *almost locally compact* if its set of points with *-compact neighborhoods, is dense. (This reduces to the usual definition of almost local compactness for topological spaces.)

Theorem 2.8. *If $\boldsymbol{X} = (X, \tau, \tau^*)$ is quasiregular and almost locally compact, then it is pseudocomplete, so (X, τ) is a Baire space. In particular, if (X, τ^*) is compact, then $\boldsymbol{X}$ is pseudocomplete, so (X, τ) is a Baire space.*

Proof. First notice there is π^0-base for (X, τ) consisting of *-closed, *-compact sets: if $T \in \tau^+$, choose $x \in T$ with a *-compact neighborhood N. By quasiregularity, there is a *-closed subset of N with nonempty interior, and it is *-compact as a *-closed subset of a *-compact set.

Let $\mathcal{B}$ be such a π^0-base for (X, τ), and let each $\mathcal{B}_n = \mathcal{B}$. Then any associated nest (B_n) is a chain of nonempty closed subsets of the compact space $(B_0, \tau^*|B_0)$, so has nonempty intersection. Thus $\boldsymbol{X}$ is pseudocomplete. $\square$

The result in Theorem 2.8 generalizes some in [3,1], in which asymmetric methods were applied early to obtain the fact that certain topological spaces are Baire spaces. In [9, 4.5] the results in [3,1] are used to show that the completely regular (in fact, 0-dimensional) space of minimal prime ideals of a commutative, reduced ring, which is rarely metrizable or locally compact, is nonetheless a Baire space.

Theorem 2.9. *Let (X, d) be a quasimetric space, and τ^* be a topology on X such that the bitopological space $\boldsymbol{X} = (X, \tau_d, \tau^*)$ is quasiregular. If each d^*-Cauchy sequence has a τ^*-limit, then $\boldsymbol{X}$ is pseudocomplete. Thus (X, τ_d) is an Oxtoby space and, consequently, a Baire space.*

Proof. Let $\mathcal{B}_n = \{B_\delta(x)\colon\ x \in X,\ 0 < \delta \leqslant 1/2^n\}$, for $n < \omega$; it suffices to show that $(\mathcal{B}_n)$ is an Oxtoby sequence; in particular, that $\bigcap_n B_n \neq \emptyset$ whenever $B_n \in \mathcal{B}_n,\ \ B_n \supset B_{n+1}$, for $n = 0, 1, \ldots.$ For such a sequence, let $B_n = B_{\delta_n}(x_n)$. For $n \leqslant m,\ x_m \in B_m \subset B_n$, so $d^*(x_m, x_n) = d(x_n, x_m) \leqslant \delta_n \leqslant 1/2^n$. Thus (x_n) is d^*-Cauchy, and so has a τ^*-limit, $x_0 \in X$. If $n \in \omega$, $m > n$, and $x_m \in B_n$, then $x_0 \in \mathrm{cl}^*\{x_m\colon\ n < m\} \subset \mathrm{cl}^* B_n \subset B_{n-1}$, thus $x_0 \in \bigcap_{n>0} B_{n-1} = \bigcap_n B_n$. $\square$

Remark 2.10. In [4], it is stated that (in different terminology) that if (X, τ_d) is a quasiregular quasimetric space in which d^*-Cauchy sequences converge, then (X, τ_d) is a Baire space. The proof of this assertion shows that (X, τ_d) is pseudocomplete in Oxtoby's original sense. As observed by Hunsaker in private correspondence, this result is similar, but incomparable with the result of Kelly given just after Proposition 2.3. Theorem 2.9 unifies these theorems, since:

Corollary 2.11.

(a) *If (X, d) is a d^*-complete quasimetric space, then (X, τ_d) is an Oxtoby space, so it is a Baire space* [14].

(b) *If (X, d) is a quasiregular quasimetric space and each d^*-Cauchy sequence converges, then (X, τ_d) is pseudocomplete, thus a Baire space* [4].

Proof. By Proposition 2.2, (X, τ_d, τ_{d^*}) is regular, thus quasiregular, so (a) follows from Theorem 2.9, with $\tau^* = \tau_{d^*}$. (b) is the special case of Theorem 2.9 in which $\tau^* = \tau_d$. $\square$

The following are examples of quasimetrics on the same set, giving regular topologies as well as regular bitopologies, which show that neither of the two classes of quasimetric spaces to which (a) and (b) above apply includes the other.

Examples 2.12.

(a) Let

$$X = [0, 1], \quad d(x, y) = \begin{cases} 1 & y < x, \\ y - x & y \geqslant x. \end{cases}$$

This quasimetric gives X the topology inherited from the Sorgenfrey line. X is d^*-complete, so part (a) of the corollary applies. Now (X, τ_d) is regular, however there are nonconstant d^*-Cauchy sequences, and none of these converges, so part (b) does not apply.

(b) Let

$$X = [0, 1], \quad d(x, y) = \begin{cases} 1 & 0 = y < x, \\ |x - y| & \text{otherwise.} \end{cases}$$

This quasimetric gives X the usual topology, but d^* gives the topological sum of $\{0\}$ and $(0, 1]$ with their usual topologies. (X, τ_d) is regular, and each d^*-Cauchy sequence is d-Cauchy as well, and so convergent. Therefore part (b) applies. Now (X, τ_d, τ_{d^*}) is regular, however $(1/n)_{n>0}$ is d^*-Cauchy, yet not d^*-convergent, so part (a) does not apply.

3. More on pseudocompleteness; product spaces and G_δ-subspaces

Oxtoby shows in [15] that products of pseudocomplete topological spaces are pseudocomplete, thus Baire. Using the characterization in Theorem 2.6, we show that this holds for pseudocomplete bitopological spaces as well. The following technical lemma aids in our discussion of products of pseudocomplete spaces.

Lemma 3.1. *If $(\mathcal{C}_n)$ is an Oxtoby sequence for a topology τ on a nonempty set X, then there is an Oxtoby sequence $(\mathcal{B}_n)$ for τ such that*:

(i) *each $\mathcal{B}_n$ contains X,*

(ii) *each $\mathcal{B}_n \setminus \{X\}$ is a subfamily of $\mathcal{C}_n$, and*

(iii) *each $\mathcal{B}_{n+1} \setminus \{X\}$ refines $\mathcal{B}_n \setminus \{X\}$ (that is, if $C \in \mathcal{B}_{n+1} \setminus \{X\}$, then $C \subset D$ for some $D \in \mathcal{B}_n \setminus \{X\}$).*

Proof. First define inductively a sequence $(\mathcal{B}_n)$ of π^0-bases such that:

(ii′) each $\mathcal{B}_n \subset \mathcal{C}_n$, and

(iii′) each $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$.

Let $\mathcal{B}_0 = \mathcal{C}_0$. Next assume that $\mathcal{B}_0, \ldots, \mathcal{B}_{m-1}$ are π^0-bases satisfying (ii′) for $n < m$ and (iii′) for $n + 1 < m$. Let $\mathcal{B}_m = \{C \in \mathcal{C}_m\colon$ there is a $B \in \mathcal{B}_{m-1}$ such that $B \supset C\}$; this satisfies (ii′) for $n = m$ and (iii′) for $n = m - 1$. Suppose $U \in \tau^+$. As $\mathcal{B}_{m-1}$ is a π^0-base, there is a $B \in \mathcal{B}_{m-1}$ with $U \supset B$; moreover, $\operatorname{int} B \neq \emptyset$. As $\mathcal{C}_m$ is a π^0-base, there is a $C \in \mathcal{C}_m$ with $B \supset \operatorname{int} B \supset C$. Thus C is in $\mathcal{B}_m$ and, as each member of $\mathcal{B}_m$ has nonempty interior, $\mathcal{B}_m$ is a π^0-base for τ. Induction provides the existence of the required sequence.

Certainly, $(\mathcal{B}_n \cup \{X\})$ is a sequence of π^0-bases satisfying (i)–(iii), and to complete the proof we need only verify that the intersection of each associated nest, (B_n), is nonempty for this new sequence. If each $B_n = X$, then $\bigcap_n B_n = X$, which is nonempty by hypothesis; otherwise, let $C_n = B_n$, for $n \geqslant m$. By (ii) and (iii), there are $C_k \in \mathcal{B}_k \setminus \{X\}$, for $k < m$, with $C_k \supset C_{k+1}$, for $k < m$. We now have $\bigcap_n B_n = \bigcap_n C_n$ which is nonempty since (C_n) is an associated nest for the Oxtoby sequence $(\mathcal{C}_n)$. □

For the next two results, recall that given a set A and for each $\alpha \in A$, a set X_α with a distinguished point $x_\alpha \in X_\alpha$, their *σ-product* is

$$\sigma_{\alpha\in A}(X_\alpha, x_\alpha) = \left\{ y \in \prod_{\alpha\in A} X_\alpha\colon \left\{\alpha \in A\colon y(\alpha) \neq x_\alpha\right\}, \text{ is countable}\right\}.$$

Sigma-products for (bi)topological spaces are given the subspace (bi)topology inherited from their products.

Theorem 3.2. *If a subspace of a product of Oxtoby spaces contains a sigma-product, then it is an Oxtoby space, thus a Baire space.*

Proof. For each $\alpha \in A$, let $(\mathcal{B}_n^\alpha)$ be an Oxtoby sequence for (X_α, τ_α), and let Y be a subspace of $\prod_A X_\alpha$ containing the σ-product $\sigma_{\alpha\in A}(X_\alpha, x_\alpha)$. The lemma allows us to assume that each $\mathcal{B}_n^\alpha$ contains X_α. For each n, define $\mathcal{C}_n$ to be $\{\prod_A B^\alpha\colon\ B^\alpha \in \mathcal{B}_n^\alpha$, for each α, and for all but a finite number of α, $B^\alpha = X_\alpha\}$, and define $\mathcal{B}_n = \{C \cap Y\colon C \in \mathcal{C}_n\}$. By construction, each $\mathcal{C}_n$ is a π^0-base for the product topology. Since the subspace Y is dense, each $\mathcal{B}_n$ is a π^0-base for the subspace topology $\tau|Y$.

To show that $(\mathcal{B}_n)$ is an Oxtoby sequence, suppose (B_n) is an associated nest. There are $B_n^\alpha \in \mathcal{B}_n^\alpha$ with each $B_n = (\prod_\alpha B_n^\alpha) \cap Y$. We show that each $B_n^\alpha \supset B_{n+1}^\alpha$: There is a countable subset A' of A such that $B_n^\alpha = X_\alpha$ for all $\alpha \in A \setminus A'$. Fix n, suppose $\beta \in A'$, and note that

$$\left(\prod_\alpha B_n^\alpha\right) \cap Y = B_n \supset B_{n+1} = \left(\prod_\alpha B_{n+1}^\alpha\right) \cap Y.$$

Let $u \in B_{n+1}^\beta$. Define $x \in Y$ by letting $x(\beta) = u$, choosing $x(\alpha)$ to be any element of B_{n+1}^α for all $\alpha \in A' \setminus \{\beta\}$, and letting $x(\alpha) = x_0(\alpha)$ for all $\alpha \in A \setminus A'$. As $\{\alpha \in A\colon x(\alpha) \neq x_0(\alpha)\} \subset A'$, x is an element of Y. Moreover, x is in $\prod_\alpha B_{n+1}^\alpha$, so it is in B_n. Now $u = x(\beta) \in B_n^\beta$, and so $B_n^\beta \supset B_{n+1}^\beta$. Therefore each $B_n^\alpha \supset B_{n+1}^\alpha$. Since each $(\mathcal{B}_n^\alpha)$ is an Oxtoby sequence, each $\bigcap_n B_n^\alpha$ is nonempty. From this we show that $\bigcap_n B_n \neq \emptyset$, so $(\mathcal{B}_n)$ is an Oxtoby sequence for $(Y, \tau|Y)$. Define x in Y as follows: For $\alpha \in A \setminus A'$, set $x(\alpha) = x_0(\alpha)$ for $\alpha \in A'$, let $x(\alpha)$ be any element of the nonempty set $\bigcap_n B_n^\alpha$. Thus x is in each $Y \cap (\prod_\alpha B_n^\alpha) = B_n$, and we are done. □

The following corollary establishes that certain subspaces of a product of pseudocomplete bitopological spaces, including the whole product, are pseudocomplete. This is known for the symmetric case [18,19].

Corollary 3.3. *Products of, and dense subspaces of, quasiregular bitopological spaces are quasiregular. Thus any subspace of a product of pseudocomplete bitopological spaces which contains a σ-product of them is pseudocomplete.*

Proof. If Y is a dense subset of (X, τ) and $T \in \tau$ is such that $T \cap Y \neq \emptyset$, then we can find $U \in \tau^+$ such that $\mathrm{cl}^* U \subset T$, and by density of Y, $U \cap Y \neq \emptyset$. For a product, suppose $x \in T \in \prod_A \tau_\alpha$. Then for some finite $\Gamma \subset A$ and for each $\alpha \in \Gamma$, there is a $U_\alpha \in \tau_\alpha$ such that $x(\alpha) \in U_\alpha$ and if $\alpha \notin \Gamma$ we set $U_\alpha = X_\alpha$, then $\prod_A U_\alpha \subset T$. So there is a $V_\alpha \in \tau_\alpha^+$ whose τ_α^*-closure is a subset of U_α; for $\alpha \notin \Gamma$ set $V_\alpha = X_\alpha$. Thus $\prod_A V_\alpha$ is a nonempty set open in the product topology, whose closure with respect to $\prod_A \tau_\alpha^*$ is contained in T, as required. Finally, by Theorem 3.2, the assertion in the second sentence follows from those in the first. □

The question of inheritance of these various completeness type properties by (dense) G_δ-subspaces is quite natural and remains unsettled for pseudocompleteness and (countable) subcompactness. (Countable subcompactness is defined for bitopological spaces in Definition 3.13 below and is discussed also in [3].) We revisit these results by considering when G_δ-subspaces of a pseudocomplete space are Baire.

We must now discuss a relation of topologies introduced in [19]. Recall that the Sorgenfrey topology σ on the reals $\mathbb{R}$ is generated by the base $\sigma_0 = \{[a, b)\colon a < b\}$ and the usual topology τ generated by $\mathcal{I} = \{(a, b)\colon a < b\}$. Each nonempty open subset of $(\mathbb{R}, \tau)$ contains a member of σ_0, and further, these topologies and bases may be interchanged: each nonempty Sorgenfrey open subset contains a nonempty member of $\mathcal{I}$.

Definition 3.4. For topologies τ and σ on a set X, $\tau \Pi \sigma$ if τ^+ is a π^0-base for σ.

This turns out to be an equivalence relation for topologies on a set, and so we will later say that τ and σ are *Π-related* if $\tau \Pi \sigma$. Proposition 3.5 implies the symmetry of this relation; reflexivity and transitivity are direct.

Proposition 3.5. *If $\mathcal{B}$ is a π^0-base for a topology σ on a set X, then $\mathcal{B}$ is a π^0-base for any topology τ on X that is Π-related to σ.*

Proof. We first show that each element of $\mathcal{B}$ has nonempty τ-interior: If B in $\mathcal{B}$, then $\operatorname{int}_\sigma B \neq \emptyset$, thus there is a V in τ^+ with $\operatorname{int}_\sigma B \supset V$, that is $\operatorname{int}_\tau B \neq \emptyset$. We complete the proof by showing that each nonempty τ-open set contains an element of $\mathcal{B}$: if $V \in \tau^+$, then since τ^+ is a π^0-base for σ, $\operatorname{int}_\sigma V \neq \emptyset$, so there is a B in $\mathcal{B}$ with $V \supset \operatorname{int}_\sigma V \supset B$. □

It is proved in [19], that if $\tau \Pi \sigma$ and (X, τ) is a Baire space, then so is (X, σ). We now notice that an Oxtoby sequence for σ is one for τ in this situation as well.

Corollary 3.6. *If $(\mathcal{B}_n)$ is an Oxtoby sequence of π^0-bases for a topology σ on a set X, then $(\mathcal{B}_n)$ is such a sequence for every Π-related topology τ on X.*

Proof. By Proposition 3.5, each $\mathcal{B}_n$ is a π^0-base for τ. It remains true that each associated nest has nonempty intersection. □

The next example shows that the Π-relation does not preserve quasiregularity.

Example 3.7. Let $X = \mathbb{Q} \cup \{\infty\}$, $\mathbb{Q}$ the set of rationals, ∞ an ideal point. Let σ and τ be topologies for X for which each point q of $\mathbb{Q}$ has $\{(a, b) \cap \mathbb{Q}\colon a, b \in \mathbb{Q},\ a < q < b\}$ as a neighborhood base; a neighborhood base at ∞ is given for σ by complements of compact subsets of $\mathbb{Q}$ and one for τ is given by complements of subsets of $\mathbb{Q}$ which are closed and bounded in the usual metric. (X, τ) is homeomorphic to a subspace of the one-point compactification of the reals $\mathbb{R}$ with the usual topology, so it is regular. Any nonempty σ-open subset contains a nonempty set of the form $(a, b) \cap \mathbb{Q}$, both τ- and σ-open, so the two topologies are Π-related. Moreover, if K is a compact subset of $\mathbb{Q}$, then $(X \setminus K) \cap (a, b) \cap \mathbb{Q} \neq \emptyset$, so ∞ is in $\operatorname{cl}_\sigma((a, b) \cap \mathbb{Q})$. Therefore (X, σ) is not quasiregular.

Definition 3.8. A subset Y of a topological space X is *present for a π^0-base $\mathcal{B}$ on X* if whenever $U \in \tau^+$, if $U \cap Y \neq \emptyset$, then there is a $B \in \mathcal{B}$ such that $B \subset U$ and $\operatorname{int}(B) \cap Y \neq \emptyset$. A subset Y of X is *present for an Oxtoby sequence* $(\mathcal{B}_n)$ if it is present for each $\mathcal{B}_n$.

Remark. The collection of present subsets depends on the π^0-base or Oxtoby sequence. For example, in the Sorgenfrey topology, $\{0\}$ is present for the π^0-base $\{[a,b)\colon a<b\}$, but not for the π^0-base $\{(a,b)\colon a<b\}$. However, open sets and dense sets are certainly always present for all π^0-bases or Oxtoby sequences. All sets are present if the π^0-base contains a neighborhood base at each point; an Oxtoby sequence $(\mathcal{B}_n)$ for which each $\mathcal{B}_n$ contains a neighborhood base at each point will be called a *basic Oxtoby sequence*. Given a collection $\mathcal{B}$ of subsets of X, its *restriction* to $A \subset X$ is

$$\mathcal{B}|A = \bigl\{B \cap A\colon\ B \in \mathcal{B},\ \operatorname{int}_X(B) \cap A \neq \emptyset\bigr\}.$$

Lemma 3.9. *Suppose (X,τ,τ^*) is a bitopological space. Let $\mathcal{B}$ be a π^0-base for τ and let $A \subset X$ be present for $\mathcal{B}$. Then $\mathcal{B}|A$ is a π^0-base for A. If further, $\mathcal{B}$ consists of *-closed sets, then the subspace A is quasiregular.*

Proof. Let $G \subset A$ be a nonempty open subset of A. Let $U \in \tau^+$ be such that $U \cap A = G$. Since A is present for $\mathcal{B}$, there is a $B \in \mathcal{B}$, $B \subset U$ and $\operatorname{int}_X(B) \cap A \neq \emptyset$. Thus $B \cap A \in \mathcal{B}|A$, and $B \cap A \subset U \cap A = G$. Thus $\mathcal{B}|A$ is a π^0-base for A, and so if $\mathcal{B}$ consists of *-closed sets for X, A is also quasiregular by comments after Definition 2.5. $\square$

Theorem 3.10. *If A is a G_δ-subspace of a topological space (X,τ) which is present for an Oxtoby sequence $(\mathcal{B}_n)$ for X, then $(\mathcal{B}_n|A)$ is an Oxtoby sequence for A. Further, if τ^* is a second topology so that (X,τ,τ^*) is quasiregular, and at least one $\mathcal{B}_i$ consists of τ^*-closed sets, then A is a pseudocomplete subspace of (X,τ,τ^*).*

Proof. To show that $(\mathcal{B}_n|A)_{n<\omega}$ is an Oxtoby sequence for A, let $(P_n)_{n<\omega}$ be a $(\mathcal{B}_n|A)_{n<\omega}$-associated nest in A, and note that each $P_n = A \cap B_n$ for some $B_n \in \mathcal{B}_n$. Since A is present for each $\mathcal{B}_n$, we can use the axiom of choice to define recursively $C_0, C_1, \ldots$ such that $C_i \in \mathcal{B}_i$, $\operatorname{int}_X(C_i) \cap A \neq \emptyset$ and $C_i \subset \operatorname{int}_X(B_j) \cap C_j$ whenever $j < i$. Then $(C_n)_{n<\omega}$ is a $(\mathcal{B}_n)$-associated nest, so $\emptyset \neq \bigcap_{n<\omega} C_n \subset \bigcap_{n<\omega} P_n$. Thus $(\mathcal{B}_n|A)$ is an Oxtoby sequence on A. Finally, if $\mathcal{B}_i$ consists of τ^*-closed sets, then $(\mathcal{B}_i|A)$ is a π^0-base for $(A,\tau|A,\tau^*|A)$. So it is quasiregular by Lemma 3.9. $\square$

Example 3.11. If we let $\mathcal{B} = \{[a,b]\colon a<b \in \mathbb{R}\}$ and $\mathcal{B}_n = \mathcal{B}$ for all $n<\omega$, then $(\mathcal{B}_n)$ is an Oxtoby sequence for the Sorgenfrey line $\mathbf{L}$ for which the hypothesis of Theorem 3.10 is satisfied (symmetrically, i.e., with $\tau^* = \tau$) for any G_δ-subspace. It follows that every G_δ-subspace of the Sorgenfrey line is pseudocomplete.

If X is a regular (bi)topological space, and $\mathcal{B}$ is a π^0-base, then the set of *-closures of the elements of $\mathcal{B}$ is also a π^0-base. Hence we have from Theorem 3.10:

Corollary 3.12. *Each G_δ-subspace of a regular (bi)topological space with a basic Oxtoby sequence is pseudocomplete.*

The following is a bitopological version of a definition in [6]:

Definition 3.13. A bitopological space (X, τ, τ^*), is countably subcompact if it is regular, and τ has a base $\mathcal{B}$ for which each strong nest $(B_n)_{n<\omega} \subset \mathcal{B}$, has nonempty intersection.

Proposition 3.14. *A regular bitopological space, (X, τ, τ^*) is countably subcompact if and only if it has a base $\mathcal{B}$, such that if $\mathcal{B}_n = \mathcal{B}$ for each $n < \omega$, then $(\mathcal{B}_n)$ is an Oxtoby sequence. Thus each countably subcompact space is pseudocomplete.*

Proof. This is immediate from the Definitions 3.13 and 2.1. □

By Corollary 3.12 and Proposition 3.14, we have:

Corollary 3.15 [16]. *Each G_δ-subspace of a countably subcompact space is a Baire space.*

The following gives a G_δ-subset of an Oxtoby space that is not present for any Oxtoby sequence.

Example 3.16. A pseudocomplete, almost locally compact and metrizable topological space with a G_δ-subspace which is not a Baire space: Let $X = \mathbb{R}^2 \backslash \{(x, 0){:}\ x \notin \mathbb{Q}\}$, with the topology induced by the usual topology of the plane. Clearly X is almost locally compact, so it is pseudocomplete. But $\{(x, 0){:}\ x \in \mathbb{Q}\}$ is a G_δ in X present for no Oxtoby sequence as it is homeomorphic to the non-Baire space $\mathbb{Q}$.

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